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On Integral Operators

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Let $f_n(z) = z/(1-z)^{n+1}$, $n \in N_o$, and $f_n^{(-1)}$ be defined such that $f_n * f_n^{(-1)} = \frac{z}{1-z}$, where $*$ denotes convolution (Hadamard product). Let f be analytic in the unit disc E. We define a new operator $I_n f = f_n^{(-1)} * f$ which is analogous to one defined by Ruscheweyh. Using this operator, the classes $M_{(n)}^*$ are defined. A function f, analytic in E, is in $M_{(n)}^*$ if and only if $I_n f$ is close-to-convex. The properties of $f \in M_{(n)}^*$ are discussed in some detail. It is shown that $M_{(n)}^* \subset M_{(n+1)}^*$ for $n \in N_o$ and for $n = 0, 1$, $M_{(n)}^*$ consists entirely of univalent functions. Closure properties of some integral operators defined on $M_{(n)}^*$ are also given. © 1999 Academic Press

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1. INTRODUCTION

Let A denote the class of analytic functions defined in the unit disc $E = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. Let S, K, S^*, and C be the subclasses of A which contain, respectively, the univalent, close-to convex, starlike and convex functions.

Let $f \in A$. Denote by D^α: $A \to A$ the operator defined by

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z), \qquad \alpha > -1,$$

which implies that

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}$$

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for $n \in N \cup \{0\} = N_o$ and the operator $(*)$ stands for the Hadamard product or convolution. We note that $D^o f(z) = f(z)$ and $D' f(z) = zf'(z)$. The operator $D^n f$ is called the Ruscheweyh derivative of nth order of f. Several classes of analytic functions, defined by using this operator, have been studied, see [1, 2, 8, 10] and others.

Analogous to $D^n f$, we here define an integral operator $I_n \colon A \to A$ as follows.

Let $f_n(z) = z/(1-z)^{n+1}$, $n \in N_o$, and let $f_n^{(-1)}$ be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{1-z}. \tag{1.1}$$

Then

$$I_n f = f_n^{(-1)} * f = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f. \tag{1.2}$$

Using (1.1), (1.2), and a well-known identity for $D^n f$, we have

$$(n+1) I_n f - n I_{n+1} f = z(I_{n+1} f)'. \tag{1.3}$$

Using hypergeometric functions ${}_2F_1$, we can write (1.2) as

$$I_n f = [z\,{}_2F_1(1, 1; n+1, z)] * f.$$

We note that $I_0 f' = zf$ and $I_1 f = f$. We now define

DEFINITION 1.1. Let $f \in A$. Then $f \in N^*_{(n)}$, $n \in N_o$ if and only if $I_n f \in S^*$ for $z \in E$. It is clear that $N^*_{(o)} = C$ and $N^*_{(1)} = S^*$. The classes $N^*_{(n)}$ have been studied in [7] and it is shown that $N^*_{(n)} \subset N^*_{(n+1)}$ for each $n \in N_o$.We here extend the classes $N^*_{(n)}$ as

DEFINITION 1.2. Let $f \in A$. The $f \in M^*_{(n)}$ for $n \in N_o$ if and only if there exists $g \in N^*_{(n)}$ such that, for $z \in E$,

$$\operatorname{Re}\left\{\frac{z(I_n f(z))'}{I_n g(z)}\right\} > 0.$$

We note that $M^*_{(1)} = K$ and $M^*_{(o)} = C^*$, where C^* is the class of quasi-convex functions first introduced and studied in [6].

2. PRELIMINARY RESULTS

In order to develop some results for the classes $M^*_{(n)}$, we need

LEMMA 2.1 [4]. *Let ω be analytic in E. If $|\omega|$ assumes its maximum value on the circle $z = r$ at a point z_o, then $z_o\,\omega'(z_o) = k\,\omega(z_o)$ where $k \geq 1$.*

LEMMA 2.2 [9]. *Let $\phi \in C$ and $g \in S^*$ in E. Then for F analytic in E with $F(0) = 1$, $\phi^* Fg/\phi^* g$ is contained in the convex hull of $F(E)$.*

LEMMA 2.3 [5]. *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\psi(u, v)$ be a complex-valued function satisfying the conditions*:

(i) *$\psi(u, v)$ is continuous in $D \subset \mathbb{C}^2$,*

(ii) *$(1, 0) \in D$ and $\psi(1, 0) > 0$,*

(iii) *$\operatorname{Re}\psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.*

If $h(z) = 1 + \sum_{k=2}^{\infty} c_k z^k$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z > 0)$ in E.

LEMMA 2.4. *Let p be analytic in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in E$. Then, for $|z| = r$, $z \in E$,*

(i) $\frac{1-r}{1+r} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1+r}{1-r}$

(ii) $|p'(z)| \leq 2 \operatorname{Re} p(z)/(1 - r^2)$.

3. MAIN RESULTS

The following is an inclusion result for the family $M^*_{(n)}$.

THEOREM 3.1. *$M^*_{(n)} \subset M^*_{(n+1)}$ for each $n \in N_o$.*
In particular $M^*_{(o)} \subset M^*_{(1)}$ which implies $C^{\alpha} \subset K$. This result is known [6].

Proof. Let $f \in M^*_{(n)}$. Then, for $z \in E$,

$$\operatorname{Re}\left\{\frac{z\big(I_n f(z)\big)'}{I_n g(z)}\right\} > 0, \quad \text{for some } g \in N^*_{(n)}.$$

Define $\omega(z)$ in E such that

$$\frac{z\big(I_{n+1}(z)\big)^*}{I_{n+1}(z)} = \frac{1 - \omega(z)}{1 + \omega(z)}, \tag{3.1}$$

where $\omega(0) = 0$ and $\omega(z) \neq -1$. We show that $|\omega(z)| < 1$. From (3.1) and the identity (1.3), we have

$$(n+1)\frac{z(I_n f(z))'}{I_{n+1}g(z)} = \frac{z(I_{n+1}g(z))'}{I_z g(z)}\left[\frac{1-\omega(z)}{1+\omega(z)}\right] + \frac{I_{n+1}g(z)}{I_n g(z)}\left\{\frac{-2z\omega'(z)}{[1+\omega(z)]^2} + n\left[\frac{1-\omega(z)}{1+\omega(z)}\right]\right\}. \tag{3.2}$$

We now apply (1.3) for the function g and since $N^*_{(n)} \subset N^*_{(n+1)}$, there exists an analytic function $\omega_1(z)$ with $\omega_1(0) = 0$ and $|\omega_1(z)| < 1$ such that

$$\frac{I_n g(z)}{I_{n+1}g(z)} = \frac{1-\omega_1(z)}{1+\omega_1(z)}. \tag{3.3}$$

Thus, from (3.2) and (3.3), we have

$$\frac{z(I_n f(z))'}{I_n g(z)} = \frac{1-\omega(g)}{1+\omega(g)} + \frac{1}{n+1}\left(\frac{1+\omega_1(z)}{1-\omega_1(z)}\right)\left(\frac{2z\omega'(z)}{[1+\omega(z)]^2}\right). \tag{3.4}$$

Suppose now that, for $z \in E$,

$$\max_{|z|\leq|z_o|}|\omega(z)| = |\omega(z_o)| = 1 \qquad (\omega(z_o) \neq -1).$$

Then, from Lemma 2.1, it follows that

$$z_o\omega'(z_o) = k\omega(z_o), \quad \text{where } k \geq 1.$$

Setting $\omega(z_o) = e^{i\theta}$ and $\omega_1(z_o) = re^{i\phi}$ in (3.4) gives us

$$\begin{aligned}
\operatorname{Re}&\left\{\frac{z_o(I_n f(z_o))'}{I_n g(z_o)}\right\} \\
&= \operatorname{Re}\left\{\frac{1}{n+1}\cdot\frac{-2k(e^{i\theta}+e^{-i\theta}+2)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|1+e^{i\theta}|^2}\right\} \\
&= \frac{-4k}{n+1}\left\{\frac{(\cos\theta+1)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|1+e^{i\theta}|^2}\right\}.
\end{aligned}$$

Hence, if $\phi = \frac{\pi}{2}$,

$$\operatorname{Re}\left\{\frac{z_o(I_n f(z_o))'}{I_n g(z_o)}\right\} < 0, \quad \text{where } g \in N_{(n)}^* \text{ and } k \geq 1.$$

This contradicts our hypothesis that $f \in M_n^*$. Thus $|\omega(z)| < 1$ and so, from (3.1), $f \in M_{(n+1)}^*$. Since it is known that $C^* \subset K$ and $f \in K$ is univalent, we note that $f \in M_{(n)}^*$ is univalent in E for $n = 0, 1$. ▮

Remark 3.1. From (1.2) and Definition 1.2, it follows that

$$f \in M_{(n)}^* \text{ if and only if } I_n f \in K. \tag{3.5}$$

THEOREM 3.2. *Let $f \in M_{(n)}^*$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then*

$$|a_k| \leq \frac{(k+n-1)!}{n!(k-1)!}, \qquad k \geq 2, n \in N_o.$$

This result is sharp.

Proof. $f \in M_{(n)}^*$ implies that $F = I_n f \in K$. Therefore $F = f_n^{(-1)} * f$ and so $f_n * F = \frac{z}{1-z} * f$.

Let $F(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Then

$$|a_k| = \frac{(n+k-1)!}{n!k!}|b_k| \leq \frac{(n+k-1)!}{n!(k-1)!} \qquad (|b_k| \leq \text{ since } F \in K),$$

and this gives us the required result. The sharpness follows if we take

$$I_n f_0(z) = \frac{z}{(1-z)^2}; \qquad f_0^{(z)} = z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} z^k.$$

▮

Special Cases. (i) $f \in M_{(0)}^* \equiv C^*$ implies $|a_k| \leq 1$ for all $k \geq 2$. This result is known [6].

(ii) For $f \in M_{(1)}^* \equiv K$, $|a_k| \leq k$ for all $k \geq 2$, see [3].

(iii) Let $n \in N_o$ and $k = 2$. Then

$$|a_2| \leq (n+1).$$

Using Theorem 3.2 and the fact that $f \in M_{(n)}^*$ is univalent for $n = 0, 1$, we can immediately prove the following covering result.

THEOREM 3.3. *Let $f \in M^*_{(n)}$, $n = 0, 1$. If B is the boundary of the image of E under f, then every point of B is at distance at least $\frac{1}{n+3}$ from the origin.*

THEOREM 3.4. *Let $f \in M^*_{(n+1)}$. Then $f \in M^*_{(n)}$ for $|z| < r_n = (1+n)/(2+\sqrt{n^2+3})$. This result is sharp.*

Proof. Since $f \in M^*_{(n+1)}$, we can write

$$\frac{z(I_{n+1}f(z))'}{I_{n+1}g(z)} = H(z), \qquad \operatorname{Re} H(z) > 0,$$

$$z \in E \quad \text{for some } g \in N^*_{(n+1)}.$$

Identity (1.3) together with some computations yields

$$\left\{\frac{z(I_n f(z))'}{I_n g(z)}\right\} = \left\{H(z) + \frac{zH'(z)}{h(z)+n}\right\},$$

where

$$h(z) = \frac{z(I_{n+1}g(z))'}{I_{n+1}g(z)} \quad \text{and} \quad \operatorname{Re} h(z) > 0 \text{ in } E.$$

Therefore

$$\begin{aligned} \operatorname{Re}\left\{\frac{z(I_n f(z))'}{I_n g(z)}\right\} &= \operatorname{Re}\left\{H(z) + \frac{zH'(z)}{h(z)+n}\right\} \\ &\geq \operatorname{Re} H(z)\left\{1 - \frac{2r}{1-r^2}\cdot\frac{1}{(1-r)/(1+r)+n}\right\} \\ &= \operatorname{Re} H(z)\left\{\frac{(1+n) - 4r + (1-n)r^2}{(1-r)^2 + n(1-r^2)}\right\}, \end{aligned}$$

where we have used Lemma 2.4.

The right-hand side of the above inequality is positive for $|z| = r < r_n[(1+n)/(2+\sqrt{n^2+3})]$.

By taking $I_n f(z) = z(1-z)^2$ we see that the radius r_n is exact. ∎

To prove our next result, we need

LEMMA 3.1. *Let $G \in N^*_{(n)}$ and let, for $0 < \lambda \leq 1$, $g \in A$ be defined by*

$$g(z) = \frac{1}{\lambda} z^{1-1/\lambda} \int_0^z t^{1/\lambda - 2} G(t)\, dt. \tag{3.6}$$

Then

$$\operatorname{Re}\left\{\frac{z\big(I_n g(z)\big)'}{I_n g(z)}\right\} > \alpha \qquad (0 < \alpha < 1),$$

where

$$\alpha = \frac{2\lambda}{\left[(\lambda - 2) + \sqrt{9\lambda^2 - 4\lambda + 4}\right]}. \tag{3.7}$$

Proof. Set $z(I_n g(z))'/I_n g(z) = (1-\alpha)p(z) + \alpha$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$.

Then, from (3.6), we have

$$\frac{z\big(I_n g(z)\big)'}{I_n g(z)} = (1-\alpha)p(z) + \alpha + \frac{\lambda(1-\alpha)zp'(z)}{1 - \lambda + \alpha\lambda + \lambda(1-\alpha)p(z)}.$$

We now form the functional $\psi(u, v)$ by taking $u = p$ and $v = zp'$. So

$$\psi(u, v) = (1-\alpha)u + \alpha + \frac{\lambda(1-\alpha)v}{1 - \lambda + \alpha\lambda + \lambda(1-\alpha)u}.$$

For

$$D = \mathbb{C} \Big\backslash \left\{-\frac{1 - \lambda + \alpha\lambda}{\lambda(1-\alpha)}\right\} \times \mathbb{C},$$

the conditions (i) and (ii) of Lemma 2.3 are clearly satisfied. We proceed to verify condition (iii).

$$\begin{aligned}
\operatorname{Re}\psi(iu_2, v_1) &= \alpha + \operatorname{Re}\frac{\lambda(1-\alpha)v_1}{1 - \lambda + \alpha\lambda + i\lambda(1-\alpha)u_2} \\
&= \frac{\alpha + \lambda(1-\alpha)(1 - \lambda + \alpha\lambda)v_1}{(1 - \lambda + \alpha\lambda)^2 + \lambda^2(1-\alpha)^2 u_2^2} \\
&\le \alpha - \left(\frac{1}{2}\right)\frac{\lambda(1-\alpha)(1 - \lambda + \alpha\lambda)\big(1 + u_2^2\big)}{(1 - \lambda + \alpha\lambda)^2 + \lambda^2(1-\alpha)^2 u_2^2} \\
&= \frac{A + Bu_2^2}{2C},
\end{aligned}$$

where

$$A = 2\alpha(1-\lambda+\alpha\lambda)^2 - \lambda(1-\alpha)(1-\lambda+\alpha\lambda),$$
$$B = 2\alpha\lambda^2(1-\alpha)^2 - \lambda(1-\alpha)(1-\lambda+\alpha\lambda),$$
$$C = (1-\lambda+\alpha\lambda)^2 + \lambda^2 u_2^2(1-\alpha)^2 > 0.$$

$\operatorname{Re}\psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain α as given by (3.7) and $B \leq 0$ gives us $0 < \alpha < 1$. ∎

THEOREM 3.5. *Let $F \in M_{(n)}^*$ and let f be defined by*

$$f(z) = \frac{1}{\lambda} z^{1-1/\lambda} \int_0^z t^{(1/\lambda)-2} F(t)\, dt \tag{3.8}$$

for $0 < \lambda \leq 1$. Then there exists a function g with

$$\operatorname{Re}\left\{\frac{z(I_n g(z))'}{I_n g(z)}\right\} > \alpha$$

such that

$$\operatorname{Re}\left\{\frac{z(I_n f(z))'}{I_n g(z)}\right\} > \beta, \quad \text{for } z \in E,$$

where α is given by (3.7) *and β $(0 \leq \beta < 1)$ is defined by* (3.14).

Proof. Let $G \in N_{(n)}^*$ and let, for $0 < \lambda \leq 1$, g be given by (3.6). Then we can write

$$I_n G(z) = (1-\lambda) I_n g(z) + \lambda z (I_n g(z))',$$

and

$$I_n F(z) = (1-\lambda) I_n f(z) + \lambda z (I_n f(z))'.$$

Set

$$\frac{z(I_n f(z))'}{I_n g(z)} = (1-\beta) h(z) + \beta, \qquad h(0) = 1. \tag{3.9}$$

We show that $\operatorname{Re} h(z) > 0$ in E.

From the above relations, we have

$$\frac{z(I_nF(z))'}{I_ng(z)} = \left\{\frac{(1-\lambda)z(I_nf(z))'}{I_ng(z)} + \frac{\lambda z[z(I_nf(z))']'}{I_ng(z)}\right\} \Big/ \left[(1-\lambda) + \frac{\lambda z(I_ng(z))'}{I_ng(z)}\right]. \tag{3.10}$$

From Lemma 3.1,

$$\frac{z(I_ng(z))'}{I_ng(z)} = (1-\alpha)p(z) + \alpha, \qquad \operatorname{Re} p(z) > 0 \quad \text{in } E. \tag{3.11}$$

From (3.9) and (3.11) we obtain

$$\frac{z[z(I_nf(z))']'}{I_ng(z)} = \{(1-\alpha)p(z) + \alpha\}\{(1-\beta)h(z) + \beta\} + (1-\beta)zh'(z). \tag{3.12}$$

Using (3.9), (3.11), and (3.12) in (3.10), we have

$$\frac{z(I_nF(z))'}{I_nG(z)} = \beta + (1-\beta)h(z) + \frac{\lambda(1-\beta)zh'(z)}{(1-\lambda) + \lambda[(1-\alpha)p(z) + \alpha]}. \tag{3.13}$$

We now define $\psi(u,v)$ by taking $u = h(z)$ and $v = zh'(z)$ in (3.13) as

$$\psi(u,v) = \beta + (1-\beta)u + \frac{\lambda(1-\beta)v}{\lambda(1-\alpha)p - \lambda(1-\alpha) + 1}.$$

It is clear that $\psi(u,v)$ satisfies conditions (i) and (ii) of Lemma 2.3. To verify condition (iii), we note that

$$\operatorname{Re} \psi(iu_2, v_1) = \beta + \frac{\lambda(1-\beta)v_1\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}}{\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2p_2^2},$$

where $p(z) = p_1 + ip_2$, p_1, p_2 being functions of x and y and $\operatorname{Re} p = p_1 > 0$ in E. By putting $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\begin{aligned}
&\operatorname{Re} \psi(iu_2, v_1) \\
&\quad \leq \beta - \frac{\lambda(1-\beta)(1+u_2^2)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1]}{2\left[\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2\right]} \\
&\quad = \frac{A + Bu_2^2}{2C},
\end{aligned}$$

where

$$\begin{aligned}
C &= \{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 > 0, \\
A &= 2\beta\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 \\
&\qquad - \lambda(1-\beta)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1],
\end{aligned}$$

and

$$B = -\lambda(1-\beta)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1].$$

We note that $\operatorname{Re} \psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain $\beta \leq \beta_A$, where

$$\beta_A = \frac{\lambda^2(1-\alpha)^2 p_2^2 + \lambda[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1]}{2\left[\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 + \{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}\right]}. \tag{3.14}$$

Also, from $B \leq 0$, we have $\beta_A < 1$ and the condition (iii) is satisfied to give $\operatorname{Re} h(z) > 0$ for $z \in E$. This completes the proof. ∎

Let, for $f \in A$ and $n = (\frac{1}{\lambda} - 1)$ in (3.8),

$$f(z) = L(F) = \frac{n+1}{z^n} \int_0^z t^{n-1} F(t)\, dt,$$

$$\begin{aligned}
L(F) &= \left(z \sum_{k=0}^{\infty} \frac{n+1}{n+k+1} z^k \right) * F = \left(z \sum_{k=0}^{\infty} \frac{(n+1)_k (1)_k}{(n+2)_k k!} \right) * F \\
&= [z\,{}_2F_1(1, n+1; n+2, z)] * F \qquad (3.15) \\
&= \frac{z}{(1-z)^{n+1}} * \left[\frac{z}{(1-z)^{n+2}} \right]^{(-1)} * F \\
&= f_n * f_{n+1}^{(-1)} * F.
\end{aligned}$$

This implies that

$$I_n L(F) = I_{n+1} F.$$

Thus we have

THEOREM 3.6. *Let $F \in M^*_{(n+1)}$. Then $f = L(F) \in M^*_{(n)}$ for each $n \in N_0$ and $f = L(F)$ is defined by* (3.15).

Next we prove that the classes $M^*_{(n)}$ are closed under convolution with convex functions. Many results on $M^*_{(n)}$ can be deduced as applications of this result.

THEOREM 3.7. *Let $\phi \in C$ and $f \in M^*_{(n)}$. The $f * \phi \in M^*_{(n)}$.*

Proof. Let

$$\begin{aligned} I_n(f * \phi) &= f_n^{(-1)} * (f * \phi) \\ &= \left(f_n^{(-1)} * f\right) * \phi \\ &= (I_n f) * \phi. \end{aligned}$$

Since $I_n f \in K$ and $\phi \in C$, it follows that $I_n f * \phi \in K$, see [9]. Consequently $f * \phi \in M^*_{(n)}$. ∎

We can apply Theorem 3.7 to note that the classes $M^*_{(n)}$ are invariant under the following operators.

(i) $f_1(z) = \int_0^z \frac{f(t)}{t}\, dt$

(ii) $f_2(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt}$, $|x| \leq 1$, $x \neq 1$

(iii) $f_3(z) = (1 + c)/z^c \int_0^t t^{c-1} f(t)\, dt$, $\operatorname{Re} c > 0$.

The proof follows immediately since we can write

$$f_i = f * \phi_i, \qquad i = 1, 2, 3,$$

with

$$\phi_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1 - z),$$

$$\phi_2(z) = \sum_{k=1}^{\infty} \frac{1 - x^k}{k(1 - x)} z^k = \frac{1}{1 - x} \log \frac{1 - xz}{1 - z} \qquad (|x| = 1,\ x \neq 1),$$

$$\phi_3(z) = \sum_{k=1}^{\infty} \frac{1 + c}{k + c} z^k, \qquad \operatorname{Re} c > 0,$$

and ϕ_i is convex for each $i = 1, 2, 3$.

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